§2 Real Numbers

2.1 Construction of $\mathbb{R}$

Construction of $\mathbb{R}$:

1) Completion of $\mathbb{Q}$.

Regard $\mathbb{Q}$ as a metric space with $d(x,y) := |x-y|$, define $\mathbb{R}$ to be the completion of $\mathbb{Q}$.

(Refer to [3])

2) Axiomatic Approach :

(Roughly speaking : impose / assume properties we need)

(i) Field axioms / Algebraic properties

Field axioms / Algebraic properties :

$(\mathbb{R}, +, \cdot, 0, 1)$ equips with additions $+$ and multiplication $\cdot$ that satisfy :

(A1) (Commutative law) $a+b=b+a$ for all $a,b\in\mathbb{R}$.

(A2) (Associative law) $(a+b)+c = a+(b+c)$ for all $a,b,c\in\mathbb{R}$

(A3) (Existence of 0) there exists $0\in\mathbb{R}$ such that $a+0=0+a$ for all $a\in\mathbb{R}$

(A4) (Existence of additive inverse) for all $a\in\mathbb{R}$, there exists $b\in\mathbb{R}$ such that

$$a+b=b+a=0$$

(M1) (Commutative law) $a\cdot b = b\cdot a$ for all $a,b\in\mathbb{R}$.

(M2) (Associative law) $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ for all $a,b,c\in\mathbb{R}$

(M3) (Existence of 1) there exists $1\in\mathbb{R}$ such that $a\cdot 1 = 1\cdot a$ for all $a\in\mathbb{R}$

(M4) (Existence of multiplicative inverse) for all $a\in\mathbb{R}\setminus\{0\}$, there exists $b\in\mathbb{R}$ such that

$$a\cdot b = b\cdot a = 1$$

(D) (Distributive law) $a\cdot(b+c) = a\cdot b + a\cdot c$ and $(b+c)\cdot a = b\cdot a + c\cdot a$

for all $a,b,c\in\mathbb{R}$

Idea : Forget everything you learned before, start from those axioms (things accepted to be true) and prove things you suspect to be true.

Theorem :

a) (Uniqueness of additive inverse)

If $b, c \in \mathbb{R}$ are additive inverses of $a \in \mathbb{R}$, then $b = c$.

(Therefore we denote it by $-a$)

b) (Uniqueness of multiplicative inverse)

If $b, c \in \mathbb{R}$ are multiplicative inverses of $a \in \mathbb{R} \setminus \{0\}$, then $b = c$.

(Therefore we denote it by a^{-1} or $\frac{1}{a}$)

proof :

(a) By assumption, $a + b = b + a = 0$ and $a + c = c + a = 0$

Now,
$$\begin{aligned} c &= 0 + c && \text{(A3)} \\ &= (b+a) + c \\ &= b + (a+c) && \text{(A2)} \\ &= b + 0 \\ &= b && \text{(A3)} \end{aligned}$$

$\therefore b = c$

(b) Exercise !

Theorem :

a) (Uniqueness of 0)

If $z \in \mathbb{R}$ such that $z + a = a$ for all $a \in \mathbb{R}$, then $z = 0$.

b) (Uniqueness of 0)

If $u \in \mathbb{R}$ such that $u \cdot a = a$ for all $a \in \mathbb{R}$, then $u = 1$.

c) If $a \in \mathbb{R}$, $a \cdot 0 = 0$

proof of (c) :

$$\begin{aligned} a + a\cdot 0 &= a \cdot 1 + a \cdot 0 \\ &= a \cdot (1 + 0) \\ &= a \cdot 1 \\ &= a \end{aligned}$$

$(-a) + a + a \cdot 0 = -a + a$

$\therefore a \cdot 0 = 0$

(Think : What do we use in each step ?)

Theorem :

If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

proof :

It suffices to show if $a \neq 0$, then $b = 0$.

By assumption ,
$$a \cdot b = 0$$
$$\frac{1}{a} \cdot (a \cdot b) = \frac{1}{a} \cdot 0$$
$$(\frac{1}{a} \cdot a) \cdot b = 0 \qquad \text{(M2 and previous theorem)}$$
$$1 \cdot b = 0 \qquad \text{(M4)}$$
$$b = 0 \qquad \text{(M3)}$$

Exercise : Show that $(-1) \cdot (-1) = 1$

Definition :

- (Subtraction)

  If $a, b \in \mathbb{R}$, $a - b$ is defined as $a + (-b)$.

- (Division)

  If $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$, $a \div b$ is defined as $a \cdot (\frac{1}{b})$.

(ii) Order properties of $\mathbb{R}$

Order properties of $\mathbb{R}$:

There is a subset $\mathbb{P} \subseteq \mathbb{R}$, called the set of positive real numbers, that satisfies

- $a, b \in \mathbb{P} \Rightarrow a + b \in \mathbb{P}$
- $a, b \in \mathbb{P} \Rightarrow a \cdot b \in \mathbb{P}$
- (Trichotomy property) If $a \in \mathbb{R}$, then exactly one of the following holds :

$$a \in \mathbb{P} \ , \ a = 0 \ , \ -a \in \mathbb{P}$$

Theorem :

$1 \in \mathbb{P}$.

proof :

By the last property and the fact that $0 \neq 1$, we have either $1 \in \mathbb{P}$ or $-1 \in \mathbb{P}$

However, if $-1 \in \mathbb{P}$, then $1 = (-1) \cdot (-1) \in \mathbb{P}$ (Contradiction)

(Why? Exercise !)

Definition :

If $a, b \in \mathbb{R}$,

- If $a-b \in \mathbb{P}$, then we write $a>b$ (a is greater than b) or $b<a$ (b is less than a).
- If $a-b \in \mathbb{P} \cup \{0\}$, then we write $a \geq b$ or $b \leq a$.

Trichotomy property can be reformulated as :

If $a, b \in \mathbb{R}$, then exactly one of the following holds :

$a-b \in \mathbb{P}$, $a-b=0$, $-(a-b) = b-a \in \mathbb{P}$

(Why? Exercise!)

i.e. $a>b$, $a=b$, $a<b$

Theorem :

Let $a, b, c \in \mathbb{R}$, then

- $a>b$ and $b>c \Rightarrow a>c$
- $a>b \Rightarrow a+c > b+c$
- $a>b$ and $c>0 \Rightarrow ca>cb$

  $a>b$ and $c<0 \Rightarrow ca<cb$

proof : Exercise!

Theorem :

a) If $a \in \mathbb{R} \setminus \{0\}$, then $a^2 = a \cdot a > 0$

b) If $n \in \mathbb{N}$, then $n>0$

proof of (b) :

Mathematical Induction.

Exercise :

1) If $a \in \mathbb{P}$, show that $\frac{1}{a} \in \mathbb{P}$.

2) If $a>1$, show that $\frac{1}{a} < 1$.

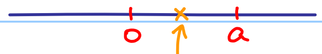
3) If $a, b \in \mathbb{P}$ and $a^2 > b^2$, show that $a>b$.

Theorem :

If $a \in \mathbb{R}$ such that $0 \le a < \varepsilon$ for every $\varepsilon > 0$, then $a = 0$.

proof :

Suppose $a > 0$.

Idea : 

take $\varepsilon = \frac{1}{2}a > 0$, then we get contradiction.

Main issues : Why $a > \frac{1}{2}a$?

$1 - 0 = 1 \in \mathbb{P}$

$1 > 0$

$2 = 1 + 1 > 0 + 1 = 1$

$2 > 1$

$1 > \frac{1}{2}$ (By previous exercise)

$a > \frac{1}{2} \cdot a$

Theorem :

If $ab > 0$, then either $a, b > 0$ or $a, b < 0$.

(iii) Completeness properties of $\mathbb{R}$

Definition :

Let S be a nonempty subset of $\mathbb{R}$.

- S is said to be bounded above (below) if there exists $u \in \mathbb{R}$ ($\ell \in \mathbb{R}$) such that $s \le u$ ($\ell \le s$) for all $s \in S$. Each u (ℓ) is called an upper bound (a lower bound) of S.
- S is said to be bounded if it is both bounded above and below.
- If S is bounded above (below), then $u \in \mathbb{R}$ ($\ell \in \mathbb{R}$) is said to be a supremum (infimum) or a least upper bound (greatest lower bound) of S if it satisfies
  (1) u (ℓ) is an upper (a lower) bound of S,
  (2) If u' (ℓ') is an upper (a lower) bound of S, then $u \le u'$ ($\ell' \le \ell$).
  We denote u and ℓ by $\sup S$ and $\inf S$ (By showing the uniqueness of them).

Lemma: (Alternative definition of supremum)

Let S be a nonempty subset of $\mathbb{R}$.

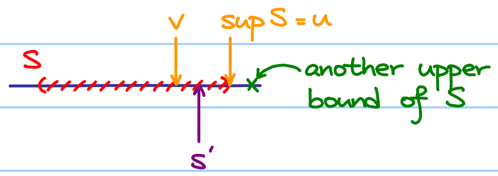
$\sup S = u$ if and only if u satisfies

(1') $s \leq u$ for all $s \in S$ (i.e. u is an upper bound)

(2') If $v < u$, then there exists $s' \in S$ such that $v < s'$

(Remark: How about $\inf S$?)

Idea:



proof:

(1) $\Leftrightarrow$ (1') trivial

(2) $\Leftrightarrow$ (2') contrapositive

Lemma: (Another alternative definition)

Let S be a nonempty subset of $\mathbb{R}$.

$\sup S = u$ if and only if u satisfies

(1') $s \leq u$ for all $s \in S$ (i.e. u is an upper bound)

(2'') for all $\varepsilon > 0$, there exists $s' \in S$ such that $u - \varepsilon < s'$

proof:

(2'') $\Leftrightarrow$ (2') : Consider $\varepsilon = u - v$ (i.e. $v = u - \varepsilon$)

Example:

Let $S = \{x \in \mathbb{R} : x < 1\}$, show that $\sup S = 1$.

(1) Clearly, 1 is an upper bound of S.

(2) Let $\varepsilon > 0$, then take $s = 1 - \frac{\varepsilon}{2} \in S$, we have $1 - \varepsilon < s$

$\therefore \sup S = 1$.

(Note $1 \notin S$, i.e. $\sup S$ is NOT necessary in S.)

Completeness properties of $\mathbb{R}$:

If S is a nonempty subset of $\mathbb{R}$ and it is bounded above, then $\sup S$ exists.

Axiomatic Construction of $\mathbb{R}$:

The set of all real numbers $\mathbb{R}$, that has (is assumed to have) the following properties :

(1) Field axioms / Algebraic properties

(2) Order properties

(3) Completeness properties

Exercise :

Prove that

1) If S is a nonempty subset of $\mathbb{R}$ and it is bounded below, then $\inf S$ exists.

(Hint : Consider $-S := \{-s : s \in S\}$)

2) If A and B are nonempty subsets of $\mathbb{R}$ and $a \le b$ for all $a \in A$ and $b \in B$,

then $\sup A \le \inf B$.

3) If S is a nonempty subset of $\mathbb{R}$ and it is bounded above, then $\sup(a+S) = \sup S + a$

where $a+S := \{a+s \in \mathbb{R} : s \in S\}$

2.2 Archimedian Property and Related Application

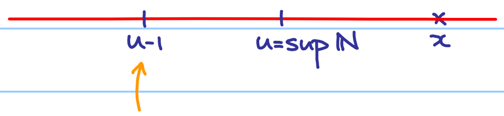
Theorem : (Archimedian property)

If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x < n_x$.

proof :

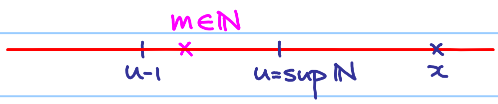
Suppose the contrary, $x \ge n$ for all $n \in \mathbb{N}$ (i.e. x is an upper bound of $\mathbb{N}$)

By completeness property, there exists $u = \sup \mathbb{N}$.



$u-1$ is NOT an upper bound

$\Downarrow$



there exists $m \in \mathbb{N}$ such that $m > u-1$

i.e. $m+1 > u = \sup \mathbb{N}$ Contradiction !

Corollary:

If $S := \{\frac{1}{n} : n \in \mathbb{N}\}$, then $\inf S = 0$

proof:

S is nonempty and bounded below by 0

By Archimedian property, for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < n$ (i.e. $\frac{1}{n} < \varepsilon$)

$$0 \le \frac{1}{n} < \varepsilon = 0 + \varepsilon$$

$\therefore \inf S = 0$.

From the proof, we can also observe that:

If $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $0 < \frac{1}{n_\varepsilon} < \varepsilon$.

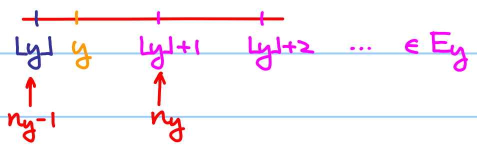
Corollary: (Refined statement of Archimedian property)

If $y > 0$, there exists $n_y \in \mathbb{N}$ such that $n_y - 1 \le y < n_y$.

proof:

Let $E_y = \{m \in \mathbb{N} : y < m\}$

Idea:



$\lfloor x \rfloor$: floor function that gives the largest integer not greater that x (i.e. $\lfloor x \rfloor \le x$)

Main issue: How to show the existence of n_y?

Archimedian property $\Rightarrow E_y \neq \phi$

Well ordering property of $\mathbb{N}$ $\Rightarrow E_y$ has a least element n_y

Then $n_y - 1 \notin E_y$ and so $n_y - 1 \le y$

Existence of m_y : By Archimedian property

Existence of n_y : By well ordering property

Theorem : (Existence of $\sqrt{2}$)

There exists a positive real number x such that $x^2 = 2$.

proof :

Let $S = \{s \in \mathbb{R} : s > 0, s^2 < 2\}$

- $1 \in S \Rightarrow S \neq \phi$
- S is bounded above by 2. (If there exists $s \in S$ such that $s > 2$, then $s^2 > 2s > 4$!)

$\therefore$ sup S exists and we define $x = \sup S$.

Claim : $x^2 = 2$.

(By Trichotomy property, we only need to show $x^2 < 2$ and $x^2 > 2$ are NOT true !)

① Suppose $x^2 > 2$ ("$\Rightarrow$" $x > \sqrt{2}$)

Idea : Show that there exists $n \in \mathbb{N}$ such that $x - \frac{1}{n}$ is another upper bound of S

(which leads contradiction)

find $n \in \mathbb{N}$ such that $\quad (x - \frac{1}{n})^2 \overset{?}{\geq} 2$

$$x^2 - \frac{2x}{n} + \frac{1}{n^2} \overset{?}{\geq} 2$$

$$(x^2 - 2) + 1 \geq (x^2 - 2) + \frac{1}{n^2} \overset{?}{\geq} \frac{2x}{n}$$

$$\frac{x^2 - 1}{2x} \overset{?}{\geq} \frac{1}{n} \quad \text{——} (*)$$

$\frac{x^2-1}{2x} > 0$, by collary of Archimedian property, $(*)$ has solution for $n \in \mathbb{N}$.

$(x - \frac{1}{n})^2 \geq 2 > s^2 \quad \forall s \in S$ and $x - \frac{1}{n}, s > 0 \Rightarrow x - \frac{1}{n} > s$ (Contradiction !)

② Suppose $x^2 < 2$ ("$\Rightarrow$" $x < \sqrt{2}$)

Exercise !

Theorem : ($\mathbb{Q}$ is dense in $\mathbb{R}$)

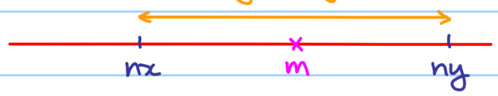
If $x, y \in \mathbb{R}$ and $x<y$, then there exists $r \in \mathbb{Q}$ such that $x<r<y$.

proof :

Without loss of generality, assume $x>0$.

Idea : length $= y-x > \frac{1}{n}$ for some $n \in \mathbb{N}$

length $= ny - nx > 1$



there exists $m \in \mathbb{N}$ such that $nx < m < ny$

$$x < \frac{m}{n} < y$$

$\overset{!!}{r} \in \mathbb{Q}$ What we need !

Similar result :

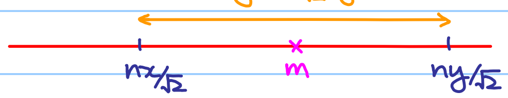
Theorem :

If $x, y \in \mathbb{R}$ and $x<y$, then there exists $p \in \mathbb{R} \setminus \mathbb{Q}$ such that $x<p<y$.

proof :

- $\sqrt{2}$ is irrational (Why?)
- length $= \frac{1}{\sqrt{2}}(y-x) > \frac{1}{n}$ for some $n \in \mathbb{N}$

length $= \frac{n}{\sqrt{2}}(y-x) > 1$



$$x < \frac{n\sqrt{2}}{m} < y$$

$\overset{!!}{p} \in \mathbb{R} \setminus \mathbb{Q}$

Same trick !

2.3 Intervals

Notations :

If $a<b$, then

$(a,b) = \{x \in \mathbb{R} : a<x<b\}$ — open interval

$[a,b] = \{x \in \mathbb{R} : a \le x \le b\}$ — closed interval

(open interval and closed interval: finite interval)

$(a,b] = \{x \in \mathbb{R} : a<x \le b\}$

$[a,b) = \{x \in \mathbb{R} : a \le x<b\}$

half open (closed) interval (finite interval)

$(a,\infty) = \{x \in \mathbb{R} : a<x\}$

$(-\infty,b) = \{x \in \mathbb{R} : x<b\}$

infinite open interval

Note : $\infty, -\infty \notin \mathbb{R}$, ONLY convention.

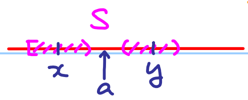
Characterization Theorem :

If $S \subseteq \mathbb{R}$ that contains at least two points and has the property

$$x,y \in S \text{ and } x<y \Rightarrow [x,y] \subseteq S \quad \text{— } (*)$$

then S is an interval.

Idea : Can S be such a subset of $\mathbb{R}$?



No ! There exists $x,y \in S$ with $x<y$ and $a \in [x,y]$ such that $a \notin S$.

Property $(*)$ governs that S must be "something nice" (an interval).

proof :

There are 4 cases :

(1) S is bounded , let $a = \inf S$ and $b = \sup S$, then $S = (a,b)$ or $[a,b]$ or $[a,b)$ or $(a,b]$

(2) S is bounded above but NOT bounded below , let $b = \sup S$, then $S = (-\infty,b)$ or $(-\infty,b]$

(3) S is bounded below but NOT bounded above , let $a = \inf S$, then $S = (a,\infty)$ or $[a,\infty)$

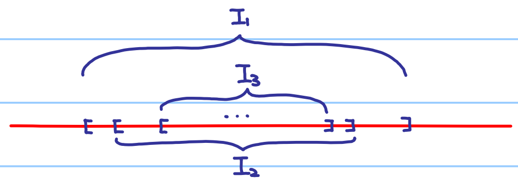
(4) S is unbounded , then $S = \mathbb{R} = (-\infty,\infty)$.

Exercise !

Definition:

A sequence of intervals I_n, $n \in \mathbb{N}$, is said to be nested if $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots$

(Here, I_n is NOT necessary to be closed)



Examples:

1) If $I_n = [0, \frac{1}{n}]$, then the sequence is nested. Furthermore $\bigcap_{n=1}^{\infty} I_n = \{0\}$.

2) If $I_n = (0, \frac{1}{n})$, then the sequence is nested. However $\bigcap_{n=1}^{\infty} I_n = \phi$.

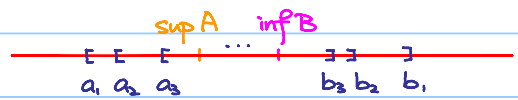
Theorem: (Nested Interval Property)

If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$, is a nested sequences of closed intervals, then there exists $p \in \mathbb{R}$ such that $p \in I_n$ for all $n \in \mathbb{N}$.

proof:

Let $A = \{a_n : n \in \mathbb{N}\}$ $\quad$ $B = \{b_n : n \in \mathbb{N}\}$

Exercise: Show that
- A is bounded above (by b_1)
- B is bounded below (by a_1)
- $a \le b$ for all $a \in A$, $b \in B$



$\therefore$ sup A and inf B exist and $\sup A \le \inf B$

Exercise: Show that for all $p \in [\sup A, \inf B]$, $p \in I_n$ for all $n \in \mathbb{N}$.

(As we can see p is NOT unique, but it is the case if we further impose a condition.)

Theorem:

Furthermore, if $\inf\{b_n - a_n = 0\}$, then there exists unique $p \in \mathbb{R}$ such that $p \in I_n$ for all $n \in \mathbb{N}$.

proof:

Suppose $p, q \in I_n$ for all $n \in I_n$ and $p \le q$, i.e. $a_n \le p \le q \le b_n$.

Then $q - p \le b_n - a_n$ for all $n \in \mathbb{N}$, i.e. $q - p$ is a lower bound of $\{b_n - a_n : n \in \mathbb{N}\}$

Therefore, $0 \le q - p \le \inf\{b_n - a_n : n \in \mathbb{N}\} = 0$

$\therefore q - p = 0$, i.e. $q = p$.

Theorem: (Uncountability of $\mathbb{R}$)

The set $\mathbb{R}$ is uncountable.

proof:

It suffices to show $[0,1]$ is uncountable.

Suppose the contrary, I is countable and $I=\{x_1, x_2, \cdots, x_n, \cdots\}$.

Construct a sequence of closed intervals:

Step 1: Choose $I_1 \subseteq I=[0,1]$ such that $x_1 \notin I_1$ (How?)

Inductive step: Choose $I_n \subseteq I$ such that $I_n \subseteq I_{n-1}$ and $x_n \notin I_n$ (How?)

By the construction, I_n is a nested sequence of intervals.

$\Rightarrow$ there exists $p \in I$ such that $p \in I_n$ for all $n \in \mathbb{N}$.

However $p \in I \Rightarrow p = x_n$ for some $n \in \mathbb{N}$

(Contradicts to $p = x_n \notin I_n$)

Binary Representation

Example:

$0.625 \in [0,1]$

$= \frac{1}{2} + \frac{1}{8}$

$= 1 \times \frac{1}{2} + 0 \times (\frac{1}{2})^2 + 1 \times (\frac{1}{2})^2 + 0 \times (\frac{1}{2})^3 + \cdots$

$\therefore$ 0.625 has a binary representation $(10100\cdots)_2$ (Remark: 0.101_2 in secondary school)

Theorem:

If $x \in [0,1]$, then there exists a sequence $\{a_n\}$ of zeros or ones such that

$$\frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{2^n} \le x \le \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_n + 1}{2^n} \quad \text{for all } n \in \mathbb{N}.$$

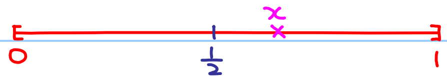
In this case, we write $x = (a_1 a_2 \cdots a_n \cdots)_2$

proof:

Idea: Bisection!

Let $x \in [0,1]$

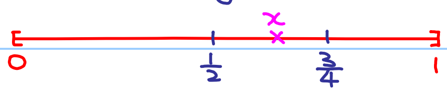
Step 1:



Let $I_1 = [0,1]$

x lies on the right subinterval of I_1, then we take $a_1 = 1$

Step 2:



Let $I_2 = [\frac{1}{2}a_1, \frac{1}{2}(a_1+1)] = [\frac{1}{2}, 1]$

x lies on the left subinterval of I_2, then we take $a_2 = 0$.

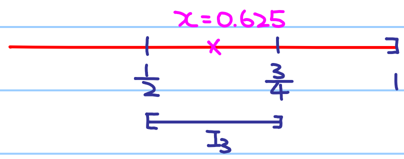
$\vdots$

Repeating $I_{n+1} = [\frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{2^n}, \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_n+1}{2^n}]$

and a_{n+1} is determined by whether x lies on the left or right subinterval of I_n.

The result follows from the fact that $x \in I_n$ for all $n \in \mathbb{N}$.

Only trouble:



$a_3 = 0$ or 1 ?

Consequence: $0.625 = (101000\cdots)_2$ OR

$(100111\cdots)_2$

$\therefore$ Binary representation is NOT unique.

However, conversely, given a sequence / representation $(a_1 a_2 a_3 \cdots)_2$, it corresponds to a unique real number in $[0,1]$.

Why? Simply because of the nested property of $\mathbb{R}$.

Exercise:

Figure out the decimal representation.

Remark: Think why $0.999\ldots = 1$?